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Finite linear spaces admitting a Ree simple group

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Abstract

This article is a contribution to the study of the automorphism groups of finite linear spaces. In particular, we look at simple groups and prove the following theorem:

Let G be a simple group and let $\mathcal{S}$ be a finite linear space on which G acts as a line-transitive automorphism group. If G is isomorphic to ${}^2G_2(q)$, then $\mathcal{S}$ is a Ree unitary space.

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1. Introduction

A *linear space* $\mathcal{S}$ is a set $\mathcal{P}$ of points, together with a set $\mathcal{L}$ of distinguished subsets called lines such that any two points lie on exactly one line. This paper will be concerned with linear spaces with an automorphism group which is transitive on the lines. This implies that every line has the same number of points and we shall call such a linear space a *regular linear space*. We shall assume that $\mathcal{P}$ is finite and that $|\mathcal{L}| > 1$.

Let G and $\mathcal{S}$ be a group and a linear space, respectively, such that G is a line-transitive automorphism group of $\mathcal{S}$. We further assume that the parameters of $\mathcal{S}$ are given by (b, v, r, k), where b is the number of lines, v is the number of points, r is the number of lines through a point and k is the number of points on a line with $k > 2$. From the assumption that G is transitive on the set $\mathcal{L}$ of lines, it follows that G is also transitive on the set $\mathcal{P}$ of points. This is a consequence of the theorem of Block given in [1].

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The line-transitive groups of automorphisms of linear spaces have been considered extensively by Camina, Praeger, Neumann, Spiezia and so on (see [3–6, 8, 12, 15, 17, 19]). In this article we prove the following theorem:

Theorem 1.1. *Let G be a simple group acting line transitively on a linear space $\mathcal{S}$. If $G \cong {}^2G_2(q)$, then $\mathcal{S}$ is a Ree Unitary space, where $q = 3^{2n+1}$ and $n \geq 1$.*

The second section describes the notation and contains several preliminary results about the group ${}^2G_2(q)$ and regular linear spaces. In the third section we give the proof of the theorem.

2. Some preliminary results

Our conventions for expressing the structure of groups run as follows. If X and Y are arbitrary finite groups, then $X \cdot Y$ denotes an extension of X by Y. The expressions $X : Y$ and $X^{\cdot}Y$ denote split and non-split extensions, respectively. The expression $X \times Y$ denotes the direct product of X and Y. The symbol $[m]$ denotes an arbitrary group of order m while Z_m or simply m denotes a cyclic group of that order. Other notation for group structure is standard. In addition, we use symbol $p^i \| n$ to denote that $p^i \mid n$ but $p^{i+1} \nmid n$ and symbol $\mathrm{Fix}_{\Omega}(K)$ to denote the set of fixed points in Ω of a subgroup K of $\mathrm{Sym}(\Omega)$. Let G be a group. We use the symbol $e(G)$ to denote the number of involutions in G.

We begin by stating some fundamental properties of ${}^2G_2(q)$, where $q = 3^{2n+1}$ and $n \geq 1$.

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [18] as subgroups of the group $GL(7, q)$. The order of ${}^2G_2(q)$ is $q^3(q^3+1)(q-1)$. Set $t = 3^{n+1}$ so that $t^2 = 3q$. We give the following information about subgroups of ${}^2G_2(q)$. For each l dividing $2n+1$, ${}^2G_2(3^l)$ denotes the subgroup of ${}^2G_2(q)$ consisting of all matrices in ${}^2G_2(q)$ with entries in the subfield of order 3^l. We use the symbols Q and K to denote a Sylow 3-subgroup and a cyclic subgroup of order $q-1$ of ${}^2G_2(q)$, respectively.

Lemma 2.1 (Theorem C of [10]). *Let $T \leq {}^2G_2(q)$ and T be maximal in ${}^2G_2(q)$. Then either T is conjugate to $P_6(l) = {}^2G_2(3^l)$ for some divisor l of $2n+1$, or T is conjugate to one of the subgroups P_i in Table 1.*

Table 1
Group conjugate to T

Group	Structure	Remarks
P_1	$Q : K$	The normaliser of Q in ${}^2G_2(q)$
P_2	$(Z_2^2 \times D_{(q+1)/2}) : Z_3$	The normaliser of a fours-group
P_3	$Z_2 \times PSL(2, q)$	An involution centraliser
P_4	$Z_{q+t+1} : Z_6$	The normaliser of Z_{q+t+1}
P_5	$Z_{q-t+1} : Z_6$	The normaliser of Z_{q-t+1}

Clearly,

$$\begin{aligned}|{}^2G_2(q)| &= q^3(q^3+1)(q-1)\\ &= q^3(q+1)(q-1)(q+1+t)(q+1-t)\\ &= 2^3 3^{3(2n+1)}\left(\frac{q-1}{2}\right)\left(\frac{q+1}{4}\right)(q+1+t)(q+1-t)\\ &= 2^3 3^{3(2n+1)}|A_0\|A_1\|A_2\|A_3|,\end{aligned}$$

where A_0, A_1, A_2, A_3 denote the cyclic subgroups of order $(q-1)/2$, $(q+1)/4$, $q+t+1$, $q-t+1$, respectively. Clearly, any two of the integers 2, 3, $(q-1)/2$, $(q+1)/4$, $(q+t+1)$, $(q-t+1)$ are relatively prime.

Lemma 2.2 (Zhou, Li and Liu [20]). *Let P be a Sylow p-subgroup of $G = {}^2G_2(q)$. We have:*

(1) *If $p = 2$, then $N_G(P) \cong Z_2^3 : Z_7 : Z_3$.*
(2) *If $p = 3$, then $N_G(P) \cong [q^3] : Z_{q-1}$.*
(3) *If $p \mid (q-1)/2$, then $N_G(P) = D_{2(q-1)}$.*
(4) *If $p \mid (q+1)/4$, then $N_G(P) = (Z_2^2 \times D_{(q+1)/2}) : Z_3$.*
(5) *If $p \mid (q+t+1)$, then $N_G(P) = Z_{q+t+1} : Z_6$.*
(6) *If $p \mid (q-t+1)$, then $N_G(P) = Z_{q-t+1} : Z_6$.*

Regarding the subgroups of the Ree group ${}^2G_2(q)$, we have the following results.

Lemma 2.3 (Levchuk and Nuzhin [11]). *A solvable subgroup $T \le {}^2G_2(q) = G$ is conjugate to a subgroup of one of the following groups: $N_G(A_i)$, $i = 0, 1, 2, 3$, $N_G(S_p)$, $p = 2, 3$, where S_p is a Sylow p-subgroup of the group ${}^2G_2(q)$.*

Lemma 2.4 (Levchuk and Nuzhin [11]). *A non-solvable subgroup $T \le {}^2G_2(q)$ is isomorphic to one of the following groups: $PSL(2,8)$, $PSL(2,q')$, $2 \times PSL(2,q')(q' > 3)$, ${}^2G_2(q')$, where q is a power of q'.*

From now on we suppose that G is an automorphism group of a linear space $\mathcal{S}$. Let G be line-transitive. Then $\mathcal{S}$ is a regular linear space. We assume that the parameters of $\mathcal{S}$ are given by (b, v, r, k) where b is the number of lines, v is the number of points, r is the number of lines through a point and k is the number of points on a line. Recall the basic counting lemmas for linear spaces.

$$vr = bk \tag{1}$$

$$v = r(k-1)+1. \tag{2}$$

Let

$$b_1 = (b, v),\ b_2 = (b, v-1),\ k_1 = (k, v),\ \text{and } k_2 = (k, v-1).$$

Obviously,

$$k = k_1k_2,\ b = b_1b_2,\ r = b_2k_2,\ \text{and } v = b_1k_1.$$

Let L be a line of $\mathcal{S}$. Then G_L will be the setwise stabilizer of L in G.

There are two facts that we are going to use throughout this article. The first is that any involution of G is conjugate to some involution of G_L. The other observation is that if an involution in G does not fix a point then G acts flag-transitively, see [7]. And so we assume that each involution fixes a point.

The following lemmas are very useful when studying the linear spaces with line-transitive automorphism groups.

Lemma 2.5 (Camina and Siemons [7]). *Let G act as a line-transitive automorphism group of a linear space $\mathcal{S}$. Let L be a line and H a subgroup of G_L. Assume that H satisfies the following two conditions:*

(i) $|\mathrm{Fix}_{\mathcal{P}}(H) \cap L| \geq 2$ *and*
(ii) *if $K \leq G_L$ and $|\mathrm{Fix}_{\mathcal{P}}(K) \cap L| \geq 2$ and K is conjugate to H in G then H is conjugate to K in G_L.*

Then either (a) $\mathrm{Fix}_{\mathcal{P}}(H) \subseteq L$ *or* (b) *the induced structure on* $\mathrm{Fix}_{\mathcal{P}}(H)$ *is also a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(H)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(H) \cap L|$. Further, $N_G(H)$ acts as a line-transitive group on this linear space.*

Lemma 2.6 (Liu, Li and Ma [16]). *Let G act as a line-transitive automorphism group of a linear space $\mathcal{S}$. Let L be a line and v even. Assume that there exists a 2-subgroup P of order 2 of G_L such that* $\mathrm{Fix}_{\mathcal{P}}(P) \subseteq L$. *Then $k \mid v$ and G is flag-transitive.*

Lemma 2.7 (Liu, Li and Ma [16]). *Let G act as a line-transitive automorphism group of a linear space $\mathcal{S}$. Let L be a line and let i be an involution of G_L. Assume that G_L has a unique conjugacy class of involutions. If*

$$|\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle) \cap L| \geq 2$$

and v is even, then G is flag-transitive or the induced structure on $\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)$ *is a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle) \cap L|$. Further, $N_G(\langle i \rangle)$ acts as a line-transitive group on this linear space.*

Lemma 2.8 (Zhou, Li and Liu [20]). *Let G act line transitively on a linear space $\mathcal{S}$. Let K be a subgroup of G. If $K \nleq G_L$ for any line $L \in \mathcal{L}$, and $K \leq G_\alpha$ for some point $\alpha \in \mathcal{P}$, then $N_G(K) \leq G_\alpha$.*

Lemma 2.9 (Liu [13] and [14]). *Let G act line transitively on a linear space $\mathcal{S}$. Assume that P is a Sylow p-subgroup of G_α for some $\alpha \in \mathcal{P}$. If P is not a Sylow p-subgroup of G, then there exists a line L through α such that $P \leq G_L$.*

The following result is useful in calculating the number of fixed points of an element.

Lemma 2.10. *Let G be a transitive group on Ω, and K be a conjugacy class of an element of G. Let $x \in K$ and* $\mathrm{Fix}_\Omega(\langle x \rangle)$ *denote the fixed points set of $\langle x \rangle$ acting on Ω. Then*

$$|Fix_\Omega(\langle x \rangle)| = |G_\alpha \cap K| \cdot |\Omega| / |K|,$$

where $\alpha \in \Omega$. In particular, if G has a unique conjugacy class of involutions, then

$$|Fix_\Omega(\langle i \rangle)| = \frac{|C_G(i)| e(G_\alpha)}{|G_\alpha|},$$

where i is an involution of G and $e(G)$ denotes the number of involutions of G.

Proof. Count the number of the order pairs (α, x), where $\alpha \in \mathrm{Fix}_{\Omega}(\langle x\rangle)$. $\square$

Consider the cycle decomposition of an involution acting on $\mathcal{P}$, we have discovered the following lemma. It is very useful in proving our theorem.

Lemma 2.11. *Let G act line transitively on a linear space $\mathcal{S}$. Let i be an involution of G_L, where L is a line of $\mathcal{S}$. Assume that i has at least two fixed points. Then*

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(\langle i\rangle)|}{|\mathrm{Fix}_{\mathcal{L}}(\langle i\rangle)|}. \tag{3}$$

Proof. Consider the cycle decomposition of i acting on $\mathcal{P}$. We know that i has $(v - |\mathrm{Fix}_{\mathcal{P}}(\langle i\rangle)|)/2$ cycles of length 2. Write $|\mathrm{Fix}_{\mathcal{L}}(\langle i\rangle)| = e$. Then i fixes e lines of $\mathcal{S}$, say L_j, where $1 \le j \le e$. Let m_j denote the number of i's 2-cycles which lie in L_j, where $1 \le j \le e$. Then

$$2\sum_{j=1}^{e} m_j = v - |\mathrm{Fix}_{\mathcal{P}}(\langle i\rangle)|.$$

Since i has at least two fixed points, we have

$$ek > 2\sum_{j=1}^{e} m_j.$$

Thus

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(\langle i\rangle)|}{|\mathrm{Fix}_{\mathcal{L}}(\langle i\rangle)|}. \quad \square$$

3. The proof of theorem

Firstly, we shall prove the following proposition.

Proposition 3.1. *Let G be a group of automorphisms of a linear space $\mathcal{S}$. Suppose that $\mathcal{S}$ is not a projective plane and $G = {}^2G_2(q)$, where $q = 3^{2n+1}$. If G is line transitive, then G is point-primitive. Further, G_α (α is a point of $\mathcal{P}$) is isomorphic to $N_G(S_3)$ or $2 \times PSL(2, q)$ or $N_G(A_i)$, where $i = 1, 2, 3$, and S_3 denotes a Sylow 3-subgroup of G.*

Proof. Since G is line-transitive and $\mathcal{S}$ is not a projective plane, we know that there exists a prime p such that $p \mid b$ but $p \nmid v$. In fact, every prime divisor p of b_2 satisfies the above condition. Thus by Lemma 2.8 we have $N_G(P) \le G_\alpha$, where P is a Sylow p-subgroup of G and $\alpha \in \mathcal{P}$. It is clear that b divides $|G| = q^3(q^3+1)(q-1)$. But then p divides $q^3(q^3+1)(q-1) = q^3(q-1)(q+1)(q+t+1)(q-t+1)$.

If $p = 3$, then $N_G(P)$ is a maximal subgroup of G. By Lemma 2.2, $G_\alpha = Q : K$ and hence G is 2-transitive on $\mathcal{P}$. By [9], we know that $\mathcal{S}$ is a Ree unitary space.

If p divides $|A_i|$, then by Lemma 2.2, $N_G(P) = N_G(A_i)$, where $i = 1, 2, 3$. Hence G_α is a maximal subgroup of G. It means that G is point-primitive.

If p divides $|A_0|$, then by Lemma 2.2, $D_{2(q-1)} \leq G_\alpha$. Since $D_{2(q-1)}$ is maximal in $2 \times PSL(2,q)$, we have $G_\alpha = D_{2(q-1)}$ or $2 \times PSL(2,q)$. If the latter occurs, then G is point-primitive. If the former occurs, then $4 \| \| G_\alpha|$. It follows that $v \equiv 2 \pmod 4$. By (1), b is odd and so $8 \| |G_L|$, where L is a line of $\mathcal{S}$. Let i be an involution in $G_L \cap G_\alpha$ and $H = \langle i \rangle$.

Suppose that G_L has a unique conjugacy class of involutions. Then by Lemma 2.7, either G is flag-transitive or there exists a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(H)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(H) \cap L|$, and $N_G(H)$ acts line transitively on this regular linear space. By Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)| = \frac{|C_G(i)|e(G)}{|G_\alpha|} = \frac{q(q^2-1)q}{2(q-1)} = q^2(q+1)/2.$$

Note that $|N_G(H)| = q(q^2-1)$, so $|\mathrm{Fix}_{\mathcal{P}}(H)|$ does not divide $|N_G(H)|$. Thus G is flag-transitive. But by [2], $v = q^3 + 1$ and so $G_\alpha = Q : K$, a contradiction.

Suppose that G_L has at least two conjugacy classes of involutions. By checking the groups in Lemmas 2.3 and 2.4, we know that G_L is isomorphic to one of the following subgroups (note that here $8 \| |G_L|$): $2 \times PSL(2,q')$, Z_2^3, $Z_2^3 : 3$, $(Z_2^2 \times D_{2u}) : Z_s$, where $q' > 3$ and u divides $(q+1)/4$ and $s \mid 3$.

If $G_L = 2 \times PSL(2,q')$, then by Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)}{q'(q'^2-1)}(q'^2 - q' + 1). \tag{4}$$

Again by (3),

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(H)|}{|\mathrm{Fix}_{\mathcal{L}}(H)|} = \frac{q(q^2+1)q'(q'^2-1)}{2(q'^2-q'+1)}.$$

On the other hand,

$$k(k-1) = |G_L|(v-1)/|G_\alpha|. \tag{5}$$

From this we can deduce that

$$k(k-1) = \frac{q'(q'^2-1)(q^2+q+1)(q^3+2)}{4}.$$

Therefore,

$$\frac{q'(q'^2-1)(q^2+q+1)(q^3+2)}{4} > \frac{q(q^2+1)q'(q'^2-1)}{2(q'^2-q'+1)}\left[\frac{q(q^2+1)q'(q'^2-1)}{2(q'^2-q'+1)} - 1\right],$$

that is

$$\frac{q(q^2+1)q'(q'^2-1)}{2(q'^2-q'+1)} < 1 + \frac{(q^2+q+1)(q^3+2)(q'^2-q'+1)}{2q(q^2+1)}.$$

This leads to

$$\begin{aligned} q &< \frac{2(q'^2 - q' + 1)}{(q^2+1)q'(q'^2-1)} + \frac{(q^2+q+1)(q^3+2)(q'^2-q'+1)^2}{q(q^2+1)^2 q'(q'^2-1)} \\ &< 1 + \frac{\left(1+\frac{1}{q}+\frac{1}{q^2}\right)\left(1+\frac{2}{q^3}\right)}{\left(1+\frac{1}{q^2}\right)^2} \times \frac{q'^2-1}{q'} \\ &< 1 + 2q', \end{aligned}$$

which forces that $q = q'$, since $q = q'^l$ for an odd positive integer l. This conflicts with $|G_L| < |G_\alpha|$.

If $G_L = Z_2^3$, then by Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{7q(q^2-1)}{8}.$$

Again by (3),

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(H)|}{|\mathrm{Fix}_{\mathcal{L}}(H)|} = \frac{4q(q^2+1)}{7}.$$

On the other hand, by (5),

$$k(k-1) = 2(q^2+q+1)(q^3+2).$$

Therefore,

$$2(q^2+q+1)(q^3+2) > \frac{4q(q^2+1)}{7}\left[\frac{4q(q^2+1)}{7} - 1\right],$$

which implies that

$$\frac{4q(q^2+1)}{7} < \frac{7(q^2+q+1)(q^3+2)}{2q(q^2+1)} + 1.$$

Consequently,

$$\begin{aligned} q &< \frac{49(q^2+q+1)(q^3+2)}{8q(q^2+1)^2} + \frac{7}{4(q^2+1)} \\ &< \frac{49 \cdot 2}{8} + 1 \\ &< 27, \end{aligned}$$

a contradiction.

If $G_L = Z_2^3 : 3$, then we can get a contradiction as above.

If $G_L = (Z_2^2 \times D_{2u}) : Z_s$, where u divides $(q+1)/4$ and $s|3$, then by Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)(4u+3)}{8us}.$$

Again by (3),

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(H)|}{|\mathrm{Fix}_{\mathcal{L}}(H)|} = \frac{4usq(q^2+1)}{4u+3}.$$

On the other hand, by (5),

$$k(k-1) = 2us(q^2+q+1)(q^3+2). \tag{6}$$

Therefore,

$$2us(q^2+q+1)(q^3+2) > \frac{4usq(q^2+1)}{4u+3}\left[\frac{4usq(q^2+1)}{4u+3} - 1\right],$$

which implies that

$$\frac{4usq(q^2+1)}{4u+3} < 1 + \frac{(4u+3)(q^2+q+1)(q^3+2)}{2q(q^2+1)}.$$

Consequently,

$$\begin{aligned} q &< \frac{4u+3}{4us(q^2+1)} + \frac{4u+3}{8u}\cdot(4u+3)\cdot\frac{q^2+q+1}{q^2+1}\cdot\frac{q^3+2}{q^3+q} \\ &< 1 + \left(\frac{1}{2}+\frac{3}{8u}\right)\cdot(q+4)\cdot\left(1+\frac{1}{q}\right) \\ &< 1 + \frac{7}{8}\cdot(q+4)\cdot\left(1+\frac{1}{27}\right) = \frac{49}{54}q + \frac{125}{27}, \end{aligned}$$

and it follows that $q < 50$. Remember that $q = 3^{2n+1}$ for some $n \geq 1$, and so we get $q = 27$. In this case, $u = 1$ or 7 and $s = 1$ or 3. However, for each of these cases, Eq. (6) has no integer solutions.

Finally, we suppose that 2 is the only prime number which divides b but does not divide v. In this case, $Z_2^3 : Z_7 : Z_3 \leq G_\alpha$ by Lemma 2.8. If $Z_2^2 \not\leq G_L$, then by Lemma 2.8 $N_G(Z_2^2) \leq G_\alpha$. Hence $G_\alpha = N_G(A_1)$ and so G is point-primitive. Therefore, we can assume that $Z_2^2 \leq G_L$. Since $8 \| |G|$, we have $b_2 = 2$. But then $|G_\alpha| = 2|G_L|$. Since $Z_2^3 : Z_7 : Z_3 \leq G_\alpha$, by Lemmas 2.3 and 2.4 we have $G_\alpha = {}^2G_2(q')$, where $q = q'^l$ and l is odd. Hence, $|G_L| = q'^3(q'^3+1)(q'-1)/2$. But by Lemmas 2.3 and 2.4, there is no subgroups of order $q'^3(q'^3+1)(q'-1)/2$.

This completed the proof of proposition. $\square$

Now we can prove our theorem stated in the introduction.

Let $H = \langle i \rangle$, i an involution in $G_\alpha \cap G_L$.

If $\mathcal{S}$ is a projective plane, then v is odd. Thus G_α contains a Sylow 2-subgroup of G and $|G_\alpha| = |G_L|$, where $\alpha \in \mathcal{P}$ and $L \in \mathcal{L}$. Thus by Lemmas 2.3 and 2.4, G_α and G_L are isomorphic to the following subgroups: $2 \times PSL(2,q')(q' > 3)$, ${}^2G_2(q')$, $Z_2^3 : K$, $(Z_2^2 \times D_{2u}) : Z_s$ and $PSL(2,8)$, where $K \leq Z_7 : Z_3$, u divides $(q+1)/4$, $s \mid 3$ and q is an odd power of q'. Hence, it is easy to see that $e(G_\alpha) \leq q^2 - q + 1$ and $e(G_L) \leq q^2 - q + 1$.

By (3), we get

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(H)|}{|\mathrm{Fix}_{\mathcal{L}}(H)|}$$
$$> \frac{q^2(q^2-q+1)}{e(G_L)} - \frac{e(G_\alpha)}{e(G_L)}.$$

Since $k(k-1) = v-1$, we have

$$v - 1 > \left[\frac{q^2(q^2-q+1)-e(G_\alpha)}{e(G_L)}\right]\left[\frac{q^2(q^2-q+1)-e(G_\alpha)}{e(G_L)} - 1\right],$$

that is

$$\frac{q(q^2-1)}{|G_\alpha|} > \frac{q^2(q^2-q+1)-e(G_\alpha)}{q^2(q^2-q+1)e(G_L)} \cdot \frac{q^2(q^2-q+1)-e(G_\alpha)-e(G_L)}{e(G_L)}. \tag{7}$$

Note that here

$$1 - \frac{e(G_\alpha)}{q^2(q^2-q+1)} \geq 1 - \frac{q^2-q+1}{q^2(q^2-q+1)} \geq 1 - \frac{1}{27^2} = a. \tag{8}$$

By (7) and (8), we get

$$\frac{q^2(q^2-q+1)}{e(G_L)} < \frac{q(q^2-1)e(G_L)}{a|G_\alpha|} + \frac{e(G_\alpha)+e(G_L)}{e(G_L)}.$$

Consequently,

$$q < \frac{(q^2-1)(e(G_L))^2}{a(q^2-q+1)|G_\alpha|} + \frac{e(G_\alpha)+e(G_L)}{q(q^2-q+1)}$$
$$< \frac{27(e(G_L))^2}{26a|G_L|} + \frac{2}{q},$$

which implies that

$$q < 1.04 \times \frac{(e(G_L))^2}{|G_L|} + 1. \tag{9}$$

When $G_L = Z_2^3 : K$ or $PSL(2,8)$, by (9), we get $q < 27$, a contradiction. When $G_L = {}^2G_2(q')$, by (9) we get $q = q'$, which conflicts with G_α being a proper subgroup of G. When $G_L = (Z_2^2 \times D_{2u}) : Z_s$, $e(G_L) = 4u + 3$, where u divides $(q+1)/4$ and $s \mid 3$. So by (9) we have

$$q < 1.04 \times \frac{(4u+3)^2}{8us} + 1 \leq 1.04 \times \frac{7(4u+3)}{8} + 1 \leq 1.04 \times \frac{7(q+4)}{8} + 1,$$

which forces that $q = 27$ and $s = 1$. In this case, $u = 1$ or 7. Therefore,

$$k(k-1) = v - 1 = \frac{27^3(27^3-1)(27-1)}{8u} - 1.$$

But this equation has no integer solutions. When $G_L = 2 \times PSL(2, q')$, where $q' > 3$, by (9) we get $q = q'$. In this case, by Lemmas 2.3 and 2.4 we have $G_\alpha = G_L = 2 \times PSL(2, q)$. But then

$$k > \frac{v - |\mathrm{Fix}_{\mathcal{P}}(H)|}{|\mathrm{Fix}_{\mathcal{L}}(H)|} = q^2 - 1,$$

which leads to

$$v - 1 = q^4 - q^3 + q^2 - 1 > (q^2 - 1)(q^2 - 2),$$

contrary to the fact that $q \geq 27$.

Therefore, we can assume that $\mathcal{S}$ is not a projective plane. But then $b_2 > 1$ and so for any prime divisor p of b_2, $N_G(P) \leq G_\alpha$, where P is a Sylow p-subgroup of G and $\alpha \in \mathcal{P}$.

By Proposition 3.1, G_α is isomorphic to $N_G(S_3)$ or $2 \times PSL(2, q)$ or $N_G(A_i)$, where $i = 1, 2, 3$. Now we divide the proof into four parts.

(1) $G_\alpha = N_G(S_3)$.

In this case, G is 2-transitive. By [9], we know that $\mathcal{S}$ is a Ree unitary space, i.e., a regular linear space with parameters $(q^2(q^2 - q + 1), q^3 + 1, q^2, q + 1)$.

(2) $G_\alpha = Z_{q+\epsilon t+1} : Z_6$, where $\epsilon = \pm 1$

In this case,

$$v = \frac{q^3(q^2 - 1)(q - \epsilon t + 1)}{6}.$$

By Lemma 2.10, we get $|\mathrm{Fix}_{\mathcal{P}}(H)| = q(q^2 - 1)/6$.

Suppose that G_L has a unique conjugacy class of involutions. Then by Lemma 2.7, either G is flag-transitive or there exists a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(H)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(H) \cap L|$, and $N_G(H)$ acts line transitively on this regular linear space. Note that b_0 divides $|N_G(H)|/2 = q(q^2 - 1)/2$ (Since here H fixes each point of $\mathrm{Fix}_{\mathcal{P}}(H)$). But $v_0 = q(q^2 - 1)/6$ and so $b_0 = v_0$ or $3v_0/2$ or $3v_0$, which conflicts with $b_0 k_0 (k_0 - 1) = v_0 (v_0 - 1)$ (Since v_0 is even). Therefore, G is flag-transitive. But by [2], this is impossible.

Suppose that G_L has at least two conjugacy classes of involutions. Checking the groups in Lemmas 2.3 and 2.4, we know that G_L is isomorphic to $2 \times PSL(2, q')$ $(q' > 3)$ or a subgroup M of $N_G(A_1)$, where q is an odd power of q' and $6 \| M|$ (note that here $6 \| G_L|$ by Lemma 2.9). When $G_L = 2 \times PSL(2, q')$, we have $e(G_L) = q'^2 - q' + 1$ and

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2 - 1)(q'^2 - q' + 1)}{q'(q'^2 - 1)}.$$

Again by (3), we know that

$$k > \frac{q'(q'^2 - 1)[q^2(q - \epsilon t + 1) - 1]}{6(q'^2 - q' + 1)}.$$

On the other hand, we have by (5)

$$k(k - 1) = \frac{q'(q'^2 - 1)}{6(q + \epsilon t + 1)} \left[\frac{q^3(q^2 - 1)(q - \epsilon t + 1)}{6} - 1 \right].$$

Therefore,

$$\frac{q'(q'^2-1)}{6(q+\epsilon t+1)}\left[\frac{q^3(q^2-1)(q-\epsilon t+1)}{6}-1\right]$$
$$> \frac{q'(q'^2-1)[q^2(q-\epsilon t+1)-1]}{6(q'^2-q'+1)}\left\{\frac{q'(q'^2-1)[q^2(q-\epsilon t+1)-1]}{6(q'^2-q'+1)}-1\right\},$$

which deduces that

$$\begin{aligned}\frac{q'(q'^2-1)[q^2(q-\epsilon t+1)-1]}{6(q'^2-q'+1)} &< 1+\frac{(q'^2-q'+1)q^3(q^2-1)(q-\epsilon t+1)}{6[q^2(q-\epsilon t+1)-1](q+\epsilon t+1)}\\ &< 1+\frac{q^3(q^2-1)(q'^2-q'+1)}{3[q^2(q-\epsilon t+1)-1]}.\end{aligned}$$

This implies that

$$\begin{aligned}q^2(q-\epsilon t+1)-1 &< 1+\frac{2q^3(q^2-1)(q'^2-q'+1)^2}{q'(q'^2-1)[q^2(q-\epsilon t+1)-1]}\\ &< 1+\frac{2(q'^2-1)}{q'}\times\frac{q^3(q^2-1)}{q^2(q-\epsilon t+1)-1}\\ &< 1+\frac{2q'q^3(q^2-1)}{q^2(q-\epsilon t+1)-1}.\end{aligned}$$

It follows that

$$\begin{aligned}q &< 1+\frac{2q'q^2(q^2-1)}{(q-\epsilon t+1)[q^2(q-\epsilon t+1)-1]}\\ &< 1+\frac{2q'}{\left(1-\frac{\epsilon t}{q}+\frac{1}{q}\right)\left(1-\frac{\epsilon t}{q}+\frac{1}{q}-\frac{1}{q^3}\right)}\\ &< 1+\frac{2q'}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)}\\ &= 1+\frac{9q'}{2}.\end{aligned}$$

This forces that $q = q'$, contradicting the fact that $|G_\alpha| > |G_L|$.

When G_L is isomorphic to a subgroup M of $N_G(A_1)$, since $6\|M|$ and M contains at least two conjugacy classes of involutions, M is isomorphic to $(Z_2^2 \times D_{2u}) : 3$ or $(Z_2 \times D_{2u}) : 3$ or $(Z_2^2 \times Z_u) : 3$, where u divides $(q+1)/4$. If $G_L \cong (Z_2^2 \times D_{2u}) : 3$, then by Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)(4u+3)}{24u}.$$

Again by (3) we get

$$k > \frac{4u[q^2(q-\epsilon t+1)-1]}{4u+3}.$$

By (5) we have

$$k(k-1) = \frac{4u}{q+\epsilon t+1}\left[\frac{q^3(q^2-1)(q-\epsilon t+1)}{6} - 1\right]. \tag{10}$$

Therefore,

$$\frac{4u[q^3(q^2-1)(q-\epsilon t+1)-6]}{6(q+\epsilon t+1)} > \frac{4u[q^2(q-\epsilon t+1)-1]}{4u+3}\left\{\frac{4u[q^2(q-\epsilon t+1)-1]}{4u+3} - 1\right\}.$$

This leads to

$$\frac{4u}{q+\epsilon t+1}\left[\frac{q^3(q^2-1)(q-\epsilon t+1)}{6} - 1\right] < 1 + \frac{(4u+3)q^3(q^2-1)(q-\epsilon t+1)}{6(q+\epsilon t+1)[q^2(q-\epsilon t+1)-1]}.$$

Note that here $u \le (q+1)/4$ and $q \ge 27$. We get

$$\begin{aligned} q-\epsilon t+1 &< 1 + \frac{(4u+3)^2}{4u} \cdot \frac{q(q^2-1)(q-\epsilon t+1)}{6(q+\epsilon t+1)[q^2(q-\epsilon t+1)-1]} \\ &< 1 + \frac{7q(q^2-1)(q+4)}{12[q^2(q-\epsilon t+1)-1]} \\ &< 1 + \frac{7q(q+4)}{12(q-\epsilon t)}. \end{aligned}$$

It follows that

$$12(q-\epsilon t)^2 < 7q^2 + 28q,$$

which forces that $q = 27$ and $\epsilon = +1$. But when $q = 27$ and $\epsilon = +1$, Eq. (10) has no the integer solutions. If $G_L = (Z_2 \times D_{2u}) : 3$, then by Lemma 2.10

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)(2u+1)}{12u}.$$

Again by (3), we get

$$k > \frac{2u[q^2(q-\epsilon t+1)-1]}{2u+1}.$$

By (5) we have

$$k(k-1) = \frac{2u}{q+\epsilon t+1}\left[\frac{q^3(q^2-1)(q-\epsilon t+1)}{6}-1\right].$$

Therefore,

$$\frac{2u}{q+\epsilon t+1}\left[\frac{q^3(q^2-1)(q-\epsilon t+1)}{6}-1\right] > \frac{2u[q^2(q-\epsilon t+1)-1]}{2u+1}\left\{\frac{2u[q^2(q-\epsilon t+1)-1]}{2u+1}-1\right\}.$$

This conflicts with the fact that $q \geq 27$. Similarly, when $G_L = (Z_2^2 \times Z_u) : 3$, we can deduce a contradiction.

(3) $G_\alpha = (Z_2^2 \times D_{(q+1)/2}) : 3$.

In this case, $v = q^3(q^2-q+1)(q-1)/6$ and

$$v-1 = (q+1)(q^5-3q^4+5q^3-6q^2+6q-6)/6.$$

We know also that $e(G_\alpha) = 4 \times (q+1)/4 + 3 = q+4$ and so

$$|\mathrm{Fix}_{\mathcal{P}}(H)| = \frac{q(q-1)(q+4)}{6}.$$

Suppose that G_L has a unique conjugacy class of involutions. Then by Lemma 2.5, either $\mathrm{Fix}_{\mathcal{P}}(H) \subseteq L$ or there exists a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(H)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(H) \cap L|$, and $N_G(H)$ act line transitively on this regular linear space. Note that $q(q-1)(q+4)/6$ does not divide $|N_G(H)| = q(q^2-1)$. Thus $\mathrm{Fix}_{\mathcal{P}}(H) \subseteq L$ and $k \geq q(q-1)(q+4)/6$. Since $|\mathrm{Fix}_{\mathcal{P}}(H)|$ is odd, k is odd. Considering the cycle decomposition of i acting on $\mathcal{P}$, we get

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{v-k}{k-1}+1 = \frac{v-1}{k-1}.$$

On the other hand, by Lemma 2.10, we have

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{|C_G(i)|}{|C_{G_L}(i)|}.$$

Therefore,

$$k-1 = \frac{(v-1)|C_{G_L}(i)|}{|C_G(i)|}. \tag{11}$$

Since $(q^5-3q^4+5q^3-6q^2+6q-6, 2q(q-1)) = 1$, we have

$$k-1 \geq (q^5-3q^4+5q^3-6q^2+6q-6)/3,$$

which conflicts with $v \geq k^2$.

Suppose that G_L has at least two conjugacy classes of involutions. Checking the groups in Lemmas 2.3 and 2.4, we know that G_L is isomorphic to $2 \times PSL(2, q')(q' \geq 27)$ or a subgroup M of $N_G(A_1)$ (note that here $3\|G_L|$ by Lemma 2.9), where q is an odd power of q' and $6\|M|$. When $G_L = 2 \times PSL(2, q')$, by Lemma 2.10, we have

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)(q'^2-q'+1)}{q'(q'^2-1)}.$$

Hence by (3), we know that

$$k > \frac{(q^4-q^3+q^2-q+4)q'(q'^2-1)}{6(q+1)(q'^2-q'+1)}.$$

It follows that

$$k > \frac{4(q^4-q^3+q^2-q+4)}{q+1},$$

which conflicts with $v \geq k^2$. Suppose that $G_L = M$, where M is a subgroup of $N_G(A_1)$ and $6\|M|$. Since M has at least two conjugacy classes of involutions, $4\|M|$. Hence M is isomorphic to one of the following subgroups: $(Z_2^2 \times D_{2u}) : 3$, $(Z_2 \times D_{2u}) : 3$, $(Z_2^2 \times Z_u) : 3$, where u divides $(q+1)/4$. If $M = (Z_2^2 \times D_{2u}) : 3$, then $e(G_L) = 4u+3$ and so by Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{q(q^2-1)(4u+3)}{24u}.$$

Hence by (3) we get

$$k > \frac{4u(q^4-q^3+q^2-q-4)}{(4u+3)(q+1)},$$

and it follows that

$$k > \frac{4(q^4-q^3+q^2-q-4)}{7(q+1)},$$

contradicting the fact that $v \geq k^2$. Similarly, for the remaining two cases, we can deduce that $v < k^2$, a contradiction.

(4) $G_\alpha = 2 \times PSL(2, q)$.

In this case, $v = q^2(q^2-q+1)$ and $v-1 = (q-1)(q^3+q+1)$. By Lemma 2.10,

$$|\mathrm{Fix}_{\mathcal{P}}(H)| = q^2-q+1.$$

Suppose that G_L has a unique conjugacy class of involutions. Then by Lemma 2.5, either $\mathrm{Fix}_{\mathcal{P}}(H) \subseteq L$ or there exists a regular linear space with parameters (b_0, v_0, r_0, k_0), where $v_0 = |\mathrm{Fix}_{\mathcal{P}}(H)|$, $k_0 = |\mathrm{Fix}_{\mathcal{P}}(H) \cap L|$, and $N_G(H)$ acts line transitively on this regular linear space. Since q^2-q+1 does not divide $q(q^2-1)$, we have $\mathrm{Fix}_{\mathcal{P}}(H) \subseteq L$. Since $|\mathrm{Fix}_{\mathcal{P}}(H)|$ is odd, k is odd. Consider the cycle decomposition of i acting on $\mathcal{P}$, we get

$$|\mathrm{Fix}_{\mathcal{L}}(H)| = \frac{v-k}{k-1} + 1 = \frac{v-1}{k-1}.$$

Note that here $(q^3 + q + 1, q(q + 1)) = 1$. Hence by (11) we get

$$k - 1 \geq q^3 + q + 1,$$

contradicting the fact that $v \geq k^2$.

Suppose that G_L has at least two conjugacy classes of involutions. By Lemma 2.9, q divides $|G_L|$. Checking the groups in Lemmas 2.3 and 2.4, we know that G_L is isomorphic to $2 \times PSL(2, q)$. Thus $b = v$, and so $\mathcal{S}$ is a projective plane, which contradicts our hypothesis.

This final contradiction finishes the proof. □

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